International Journal of Theoretical Physics, Vol. 31, No. 6, 1992

Independence of Automorphism Group, Center, and State Space of Quantum Logics

Mirko Navara[1]

Received March 2, 1990

We prove that quantum logics (=orthomodular posets) admit full independence of the attributes important within the foundations of quantum mechanics. Namely, we present the construction of quantum logics with given sublogics (=physical subsystems), automorphism groups, centers (="classical parts" of the systems), and state spaces. Thus, all these "parameters" are independent. Our result is rooted in the line of investigation carried out by Greechie; Kallus and Trnková; Kalmbach; and Navara and Pták; and considerably enriches the known algebraic methods in orthomodular posets.

1. BASIC NOTIONS AND THE MAIN RESULT

Let us first recall some basic notions; for more detail, see, e.g., Gudder (1979) and Kalmbach (1983). By a (quantum) *logic* we mean an orthomodular poset. If it is a lattice, we call it a *lattice logic.* Throughout this paper, let us reserve the symbol L for logics. Two elements $a, b \in L$ are called *orthogonal* (abbr. $a \perp b$) if $a \leq b'$. If $K \subset L$ and K is closed under the formation of orthocomplements and orthogonal joins in L, we call K a *sublogic* of L or, alternatively, we call L an *enlargement* of K. An *automorphism* of L is a bijection $\alpha: L \to L$ such that both α and α^{-1} preserve the orthocomplements and the partial ordering (thus, they preserve also all joins and meets which exist in L). The automorphism group of L is denoted by $\mathscr{A}(L)$. Two elements of L are called *compatible* if they are contained in a Boolean subalgebra of L. By $\mathscr{C}(L)$ we denote the *center* of L, i.e., the set $\{c \in L$: c is compatible to all $d \in L\}$. By a *state* on L we mean a real-valued function $s: L \to [0, 1]$ such that $s(1) = 1$ and $s(a \vee b) = s(a) + s(b)$ for any orthogonal pair $a, b \in L$. The set of all states on L is denoted by $\mathscr{S}(L)$.

[1]Department of Mathematics, Faculty of Electrical Engineering, Technical University of Prague, 166 27 Prague 6, Czechoslovakia.

The automorphism group, the center, and the state space are physically meaningful and important attributes of the logic (see, e.g., Gudder, 1979). Previous investigations led to the following results. The automorphism group of a logic can be an arbitrary group (Kalmbach, 1984; Kallus and Trnková, 1987). The center of a logic is a Boolean algebra [for further results, see, e.g., Brabec and Pták (1982) and Gudder (1979)]. The state space can be, up to an affine homeomorphism, an arbitrary compact convex subset of a locally convex topological linear space (Shultz, 1974). Besides their physical meaning, these results brought new construction principles into the study of quantum logics. It is natural to ask whether the structure of a logic induces some kind of dependence between these attributes. The independence of the center and the state space was proved in Pták (1983). [However, it should be noted that in special classes of logics—for instance, in the logics of projections in von Neumann algebras—the state space determines the center uniquely (Binder, 1986). Further, Kallus and Trnková (1987) exhibit a construction of logics with given automorphism groups and given atomistic sublogics. Navara *et al.* (1988) present a construction of logics with given centers, state spaces, and sublogics. Here we prove the following strengthening of the latter results.

Main Theorem. Suppose that K is a logic admitting at least one state, G is a group, C is a compact convex subset of a LCTLS, and B is a Boolean algebra.

Then there is a logic L such that K is a sublogic of L, the group of automorphisms of L is isomorphic to G, the state space of L is affinely homeomorphic to C, and the center of L is Boolean isomorphic to B.

The proof in the following sections is nontrivial and quite technically involved. Though I have striven for self-containedness of the proof, the reader familiar with Kallus and Trnková (1987), Navara *et al.* (1988), and Navara and Rogalewicz (1991) will find its reading more comfortable. Also, the verification of details which is left to the reader in some places will be easier if one has gone through the cited papers.

2. NOTIONS AND BASIC TECHNICAL TOOLS

Prior to the proof of our result, let us recall some more notions and facts (Gudder, 1979; Kalmbach, 1983). A maximal Boolean subalgebra B of L is called a *block*; if B is isomorphic to 2^n for $n \in N$, we call it a 2^n-block. A logic is called *chain-finite* if all its blocks are finite. The logic $\{0, 1\}$ is called *trivial.* If $a, b \in L$, $a \leq b$, we put $[a, b]_L = \{c \in L\colon a \leq c \leq b\}$ and call the set $[a, b]_L$ an *interval* in L. The interval $[0, b]_L$ becomes a logic if it is endowed with the partial ordering inherited from L (Kalmbach, 1983). An element a of L is called an *atom* if $[0, a]_L = \{0, a\}$. To avoid confusion, we sometimes indicate by indices which logic we refer to, e.g., 0_L, 1_L, $\leq_L$, ${}'^L$, $\wedge_L$, $\vee_L$, $\perp_L$.

Recall now basic constructions with logics.

Given a collection $\mathcal{L}=\{L_\alpha : \alpha \in I\}$ of logics, we call the Cartesian product $L=\prod_{\alpha\in I} L_\alpha$ a *product* of $\mathcal{L}$ if it is endowed with the "pointwise" partial ordering and orthocomplementation, i.e., for all a, $b \in L$ we have $a \leq_L b$ (resp. $a = b'^L$) iff $a(\alpha) \leq_{L_\alpha} b(\alpha)$ [resp. $a(\alpha) = b(\alpha)'^{L_\alpha}$] for all $\alpha \in I$.

Another useful construction is the pasting.

Definition 2.1 (Navara and Rogalewicz, 1991). Let $\mathcal{L}$ be a collection of logics such that for each P, $Q \in \mathcal{L}$ the intersection $P \cap Q$ is a sublogic of both P and Q and, moreover, the orthocomplementations and the partial orderings coincide on $P \cap Q$. Put $L = \bigcup_{P\in\mathcal{L}} P$ and define the binary relation $\leq_L$ and the unary operation ${}'^L$ as follows:

$$a \leq_L b \text{ (resp. } a = b'^L\text{) iff } a \leq_P b \text{ (resp. } a = b'^P\text{) for some } P \in \mathcal{L}$$

The set L equipped with $\leq_L$, ${}'^L$ is called the pasting of the collection $\mathcal{L}$.

The pasting need not be a logic in general. We recall a sufficient condition for this to hold.

Theorem 2.2 (Navara and Rogalewicz, 1991, Propositions 4.1 and 4.2). Let L be the pasting of a collection $\mathcal{L}$ of logics which satisfies the following conditions:

(L1) $\forall P, Q \in \mathcal{L}\, \exists a \in P \cap Q$: $P \cap Q = [0, a]_P \cup [a', 1]_P$.
(T2) $\exists W \in \mathcal{L}\, \forall P, Q \in \mathcal{L}, P \neq Q$: $P \cap Q \subset W$.

Then L is a logic and each block of L is a block of some logic of the collection $\mathcal{L}$.

In the definition of the pasting we have supposed that the logics of the collection $\mathcal{L}$ already have some common elements (at least 0 and 1). Alternatively, we can start with a collection of disjoint logics and form the pasting after identifying the elements of appropriate isomorphic sublogics. We shall often deal with the following two special cases:

1. In a collection $\mathcal{L}$ of disjoint logics we identify all zeros and also all units. The pasting of these logics is then called the *horizontal sum* of the collection $\mathcal{L}$ (Kalmbach, 1983). Let us call a logic *irreducible* if it cannot be expressed as a horizontal sum of nontrivial logics. Obviously, each nontrivial logic L admits a decomposition to a horizontal sum of nontrivial irreducible logics called the *summands* of L.

2. In the collection $\mathcal{L}$ of disjoint logics we identify not only all zeros and units, but also some atoms of different logics of $\mathcal{L}$ (and, necessarily, also their orthocomplements). Then we form the pasting. We say that we have *pasted* the collection $\mathcal{L}$ *by identifying* the respective *atoms*.

Theorem 2.3 (Navara and Rogalewicz, 1991, Theorem 6.1). Let K, L be logics. Suppose that a is an atom in K. Write $b = a'^K$. Put $M = [0, b]_K \times L$. For all $c \in [0, b]_K$, let us identify $c \in K$ with $(c, 0_L) \in M$ and $c \vee_K a \in K$ with $(c, 1_L) \in M$. The pasting P of K and M is then a logic. We say that P originated by the substitution of the atom a in K with the logic L.

3. PROOF OF THE MAIN THEOREM

First we prove some special cases and then we shall use them in the proof of the general case. We construct *stateless*, resp. *single-state* logics (i.e., logics whose state spaces are empty, resp. singletons) and *rigid* logics (i.e., logics with no automorphisms different from the identity).

Lemma 3.1. Let K be a chain-finite lattice logic. Then there is a proper class $\mathcal{P}$ of enlargements of K such that each $L \in \mathcal{P}$ satisfies the following two conditions:

(A1) L is rigid
(A2) L is irreducible and chain-finite.

Moreover, we may ensure that all the blocks of L which are not in K are of the form 2^3.

Proof. See Kallus and Trnková (1987). The ideas of the proof will appear (in a modified and stronger form) also in the proof of Lemmas 3.3 and 3.4. ■

Corollary 3.2. There is a proper class $\mathcal{P}_0$ of stateless logics which satisfy the conditions (A1), (A2). There is also a proper class $\mathcal{P}_1$ of logics which satisfy (A1), (A2) and admit exactly one state. Moreover, this (unique) state is two-valued. In addition, we may ensure that the logics of $\mathcal{P}_0$ have only 2^3-blocks and 2^4-blocks, and the logics of $\mathcal{P}_1$ have only 2^4-blocks and 2^5-blocks.

Proof. To obtain $\mathcal{P}_0$, one takes the finite stateless lattice logic constructed in Greechie (1971). Let us call it K and apply Lemma 3.1. The product of a stateless logic and the trivial logic is a single-state logic (Pták, 1987). Thus, the class of logics of the form $L \times \{0, 1\}, L \in \mathcal{P}_0$, can be taken for $\mathcal{P}_1$. ■

We call the blocks of the form 2^n *small blocks* if $n \leq 5$; all other blocks are called *large blocks*. The following lemma yields single-state logics with given automorphism groups.

Lemma 3.3. Let G be a group. Then there is a proper class $\mathcal{P}$ of logics such that each $L \in \mathcal{P}$ satisfies the following four conditions:

1. $\mathcal{A}(L) \cong G$.
2. card $\mathcal{S}(L) = 1$.
3. L is irreducible and has only small blocks.
4. for any $a \in L - \{0\}$ and any $\alpha \in \mathcal{A}(L)$ we have $a = \alpha(a)$ or $a \vee \alpha(a) = 1$.

Proof. We start with the logic D_{16} corresponding to the Greechie diagram in Figure 1a (Greechie, 1971). Applying Corollary 3.2, we take three nonisomorphic finite logics L_1, L_2, L_3 which are rigid and single-state. For $i = 1, 2, 3$, let us denote by d_i the (only) atom on which the state on L_i attains 1. Further, let us paste together L_1, L_2, L_3, and D_{16} by identifying the atoms a_i with d_i for $i = 1, 2, 3$—see Figure 1b. We obtain a rigid logic M with exactly one state (attaining the value 1 on a_1, a_2, a_3, and the value 0 on b, c).

According to Sabidussi (1960), there is a proper class of directed connected graphs whose automorphism groups are isomorphic to G. Sabidussi's original result is formulated for undirected graphs. If we replace each undirected edge with two directed edges, we obtain the latter result. Let (V, E) be such a graph. Replacing each edge (b, c) of E with a copy of the Greechie diagram of M (Figures 1b and 1c) we obtain the Greechie diagram of the logic L. More precisely, for each $(u, v) \in E$ we take a copy $M_{u,v}$ of M with atoms $b_{u,v}$, $c_{u,v} \in M_{u,v}$ corresponding to $b, c \in M$. We paste the collection $\{M_{u,v}\colon (u, v) \in E\}$ by identifying the following pairs of atoms:

$b_{u,v}$ with $b_{u,w}$ for all $(u, v), (u, w) \in E$
$c_{u,v}$ with $c_{t,v}$ for all $(u, v), (t, v) \in E$
$b_{u,v}$ with $c_{t,u}$ for all $(u, v), (t, u) \in E$

The pasting is the desired logic L. Indeed, the automorphism group did not change by this procedure. Further, the property (4) follows from

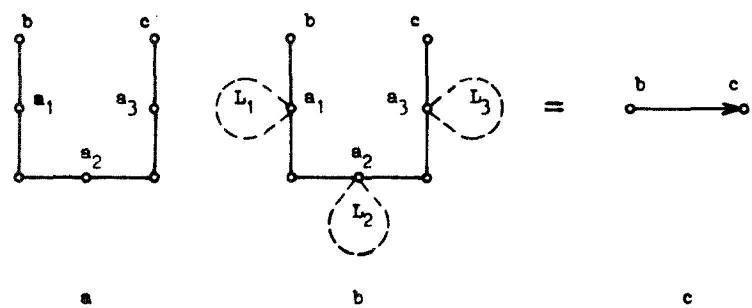

Fig. 1

the fact that for each block A of L and each $\alpha \in \mathscr{A}(L)$ we have either $A = \alpha(A)$ or $A \cap \alpha(A) = \{0, 1\}$. The remaining assertions can be verified easily. ■

We are now ready to construct rigid logics with given state spaces.

Lemma 3.4. Let K be a chain-finite logic. Then there is a proper class $\mathscr{P}$ of logics such that each $L \in \mathscr{P}$ satisfies the following four conditions:

1. K is a sublogic of L.
2. card $\mathscr{A}(L) = 1$.
3. Each state on K has a unique extension to L.
4. L is irreducible.

Proof. Without any loss of generality, we may suppose that K has only large blocks. This can be achieved by the substitution of appropriate atoms with single-state logics (Theorem 2.3). The resulting chain-finite logic has the same state space as K and K is its sublogic.

Denote by A the set of all atoms of K. Corollary 3.2 enables us to find a single-state rigid logic M. Moreover, M can be chosen so large that there is a set $\mathscr{D} = \{d_a : a \in A\}$ of atoms in M with the following properties:

1. The state on M vanishes at all elements of $\mathscr{D}$.
2. $\mathscr{D}$ contains no orthogonal pair.

For all $a \in A$, let us denote by P_a the copy of the Boolean algebra 2^3 with atoms a, d_a, and e_a $[=(a \vee d_a)']$. We take for L the pasting of the logics K, M, and P_a, $a \in A$ (of course, we first identify the zeros and units of these logics).

Every automorphism of L maps large blocks onto large blocks. Hence, it maps K onto K. It also maps $\bigcup_{a \in A} P_a$ onto itself (because P_a are the only 2^3-blocks whose one atom is contained in a large block) and M onto M. Any automorphism of L has to coincide with the identity on M and hence also on $\mathscr{D}$ and on $\bigcup_{a \in A} P_a$. So it has to coincide with the identity also on K and on the whole L. The remaining assertions are easy to prove. ■

While the pasting by identifying pairs of atoms applies well to chain-finite logics, we shall need a more general pasting technique for arbitrary logics.

Lemma 3.5. Let K be a logic with $\mathscr{S}(K) \neq \varnothing$. Then there is a logic L such that the following three conditions are fulfilled:

1. K is a sublogic of L.
2. card $\mathscr{A}(L) = 1$.
3. card $\mathscr{S}(L) = 1$.

Proof. As in the proof of Lemma 3.4, we may assume that K has only large blocks. We take two stateless logics of Corollary 3.2 such that they are nonisomorphic to each other and nonisomorphic to any interval $[0, a]_K (a \in K)$. Their product constitutes a stateless rigid logic P having only large blocks.

According to Marlow (1978), we can choose an M-base $\mathcal{M}$ in K. ($\mathcal{M}$ is a maximal subset of K containing no orthogonal pair.) It is known that $K = \mathcal{M} \cup \mathcal{M}'$, where $\mathcal{M}' = \{b' \colon b \in \mathcal{M}\}$. For each $a \in \mathcal{M}'$, let us take a logic $Q_a = [0, a]_K \times P \times \{0, 1\}$. We now identify each $b \in [0, a]_K$ with $(b, 0, 0) \in Q_a$ and we also identify the respective orthocomplements. We denote by e_a the atom $(0, 0, 1) \in Q_a$ and by $\mathscr{E}$ the set $\{e_a \colon a \in \mathcal{M}'\}$.

According to Theorem 2.2, the pasting $M = K \cup \bigcup_{a \in \mathcal{M}'} Q_a$ is a logic. [This step is described in more detail in Navara *et al.* (1988). The requirement that $\mathcal{M}$ is an M-base is used to satisfy the condition (L1) in Theorem 2.2.] Notice that M has only large blocks. Each state on K has a unique extension to M. On the other hand, each state on M is uniquely determined by its values on $\mathscr{E}$, because P is stateless and $\mathcal{M} \cup \mathcal{M}' = K$.

We shall prove that the only automorphism of M which coincides with the identity on $\mathscr{E}$ is the identity on M. If an automorphism preserves some $e_a \in \mathscr{E}$, it maps Q_a onto itself (because Q_a is the set of all elements of M compatible with e_a and the compatibility is preserved by all automorphisms). According to the construction of P, it maps $[0, a]_K$ (as a subset of Q_a) onto itself (because it maps factors of a product onto factors) and a onto a. So the automorphism coincides with the identity on $\mathcal{M}'$. Hence it coincides with the identity on the whole K and on M as a consequence.

We shall now construct a collection $\{U_a \colon a \in \mathcal{M}'\}$ of mutually nonisomorphic single-state logics with further special properties. We take a single-state logic S (from Corollary 3.2) with an atom b on which the state attains the value 1. We take the logic H_3 of projections in a 3-dimensional Hilbert space and we paste S with H_3 by identifying b with an atom of H_3. The resulted logic is denoted by U. According to Gleason's theorem (Gleason, 1957), U is a single-state logic. As in Lemma 3.4, U can be enlarged to a rigid single-state irreducible logic. We can easily modify the proof of Lemma 3.4 so that the blocks of H_3 remain 2^3-blocks. (We omit the substitution at the very beginning of the proof which led to large blocks. All arguments concerning the automorphism group of the resulting logic remain valid. The only 2^3-blocks are the blocks of H_3 and the blocks which are denoted by P_a in the proof of Lemma 3.4. The latter have only one atom common with other 2^3-blocks, so they are distinguished from the blocks of H_3 by a property which is preserved by all automorphisms.) Again, there is a proper class of such extensions. Thus we can find a family of mutually disjoint and nonisomorphic enlargements $\{U_a \colon a \in \mathcal{M}'\}$ of U which are rigid, single-state, and irreducible. Moreover, U_a contains only small blocks. For any real number $r \in [0, 1]$ there is an atom of U_a on which the state on U_a attains the value r.

We fix a state s on M and for each $a \in \mathcal{M}'$ we choose an atom $u_a \in U_a$ on which the state on U_a attains the value $s(e_a)$. Then we paste M with U_a $(a \in \mathcal{M}')$ by identifying the atoms e_a with u_a. We obtain a logic L.

Each state on L coincides with s on $\mathscr{E}$ and hence it coincides also on M. The extension of s to all U_a $(a \in \mathcal{M}')$ is possible and unique and it gives the only state on L. All small blocks of L are exactly the blocks of the logics $U_a(a \in \mathcal{M}')$. The Greechie diagram of U_a is connected, but U_a and U_b (for $a \neq b$) are not connected by small blocks. Hence, each automorphism maps each U_a $(a \in \mathcal{M}')$ onto some $U_{f(a)}$, where $f(a) \in \mathcal{M}'$. Since the logics U_a $(a \in \mathcal{M}')$ are rigid and mutually nonisomorphic, we have $f(a) = a$ and each automorphism of L must coincide with the identity on $\bigcup_{a \in \mathcal{M}'} U_a$. But this union contains also the set $\mathscr{E}$ and so L is rigid. ■

The Proof of the Main Theorem. We start with the given logic K. According to Lemma 3.5, there is a rigid single-state enlargement K_1 of K. Lemma 3.3 gives a single-state logic K_2 with $\mathscr{A}(K_2) \cong G$. According to Shultz (1974), there is a chain-finite logic whose state space is affinely homeomorphic to C. Applying Lemma 3.4, we obtain a rigid logic K_3 with $\mathscr{S}(K_3)$ affinely homeomorphic to C. The logics K_2, K_3 are irreducible and we can choose them (from the proper classes of such logics) so that they are nonisomorphic to each other and nonisomorphic to any summand of K_1.

Consider the horizontal sum H of K_1, K_2, and K_3. It contains K as a sublogic. Moreover, $\mathscr{S}(H)$ is affinely homeomorphic to C. The automorphisms of H map summands onto (the same) summands and they may differ from the identity only on the summand K_2. Hence, $\mathscr{A}(H) \cong \mathscr{A}(K_2) \cong G$. The center of H is $\{0, 1\}$.

By the application of Lemma 3.5 we obtain a rigid enlargement M of H.

Without any loss of generality we may assume that the Boolean algebra B is represented by clopen subsets of its Stone space X. Let us fix one point $z \in X$. Denote by I the ideal $\{A \in B: z \notin A\}$. According to Corollary 3.2, we can find a collection of irreducible stateless rigid logics $\{P_A: A \in I\}$ which are nonisomorphic to each other and to any summand of M. For all $x \in X$ we define the logic Q_x as follows: We put $Q_z = H$ and for Q_x $(x \neq z)$ we take the horizontal sum $M \cup \bigcup\{P_A: A \in I, x \in A\}$. We define a subset L_0 of the product $Q = \prod_{x \in X} Q_x$ consisting of all functions $f \in Q$ such that the range $f(X)$ is finite and f is measurable with respect to B [i.e., $f^{-1}(a) \in B$ for all $a \in f(X)$]. To prove that L_0 is a sublogic of Q, we need to verify that it is closed under the formation of orthocomplements, which is obvious, and that it is closed under the formation of orthogonal joins in Q. Suppose that f, $g \in L_0$ and $f \perp g$. The collection $\{f^{-1}(c) \cap g^{-1}(d)$: $c \in f(X)$, $d \in g(X)\} \subset B$ is a finite covering of X by sets on which both f and g are constant. We need to prove that $c \vee_{Q_x} d$ (where $c \perp_{Q_x} d$) does not depend on the choice of $x \in f^{-1}(c) \cap g^{-1}(d)$. The case when $c = 0$ or $d = 0$ is trivial. If $c \neq 0 \neq d$, the orthogonality of c and d implies that they belong to the same summand, say S, of Q_x. Then $c \vee_{Q_x} d = c \vee_S d$ in each logic Q_x containing $\{c, d\}$ [in particular, for $x \in f^{-1}(c) \cap g^{-1}(d)$]. Hence, $f \vee_Q g \in L_0$ and L_0 is therefore a logic.

We take for L the set of all functions $h \in Q$ for which there is an $f \in L_0$ and $\alpha \in \mathscr{A}(H)$ (recall that $H = Q_z$) satisfying $h(x) = f(x)$ for $x \neq z$ and $h(z) = \alpha f(z)$. To prove that L is a logic, it again suffices to show that L is closed under the orthogonal joins in Q. If $\{z\} \in B$, then $L = L_0$. Suppose that $\{z\} \notin B$ and h, $k \in L$, $h \perp k$. Denote by f (resp. g) the element of L_0 coinciding with h (resp. k) on $X - \{z\}$ and by α (resp. β) the automorphism of H satisfying $h(z) = \alpha f(z)$ [resp. $k(z) = \beta g(z)$]. We put $a = f(z)$ and $b = g(z)$. The set $f^{-1}(a) \cap g^{-1}(b) \in B$ contains a point different from z and hence $a \perp_H b$. This implies $f \perp_Q g$. We need to find an automorphism of H mapping $(f \vee_Q g)(z) = a \vee_H b$ onto $(h \vee_Q k)(z) = \alpha(a) \vee_H \beta(b)$. We claim that at least one of the automorphisms α, β has this property. The case when $a = 0$ or $b = 0$ is trivial. Suppose therefore that $a \neq 0 \neq b$. We have $\beta(b) \perp_H \alpha(a) \perp_H \alpha(b)$. The nonzero elements $\alpha(b)$ and $\beta(b)$ have a common upper bound $(\alpha(a))' \neq 1$. The automorphism $\beta \circ \alpha^{-1}$ maps $\alpha(b)$ onto $\beta(b)$. This, together with Lemma 3.3(4), gives that $\alpha(b) = \beta(b)$ and $\alpha(a) \vee_H \beta(b) = \alpha(a \vee_H b)$. We have proved that L is a logic. The logic K is isomorphic to the sublogic of all constant functions from X to K.

All logics Q_x, $x \in X$, have trivial centers. It is easily seen that each function $f \in L$ attaining a value different from 0 and 1 is noncompatible with some other element of L. Thus, $\mathscr{C}(L)$ contains only the characteristic functions of the sets of B and $\mathscr{C}(L) \cong B$.

We shall now prove that $\mathscr{S}(L)$ is affinely homeomorphic to $\mathscr{S}(Q_z)$ [$= \mathscr{S}(H)$]. Suppose that $A \in I$. Denote by $h_A \in L$ the characteristic function of A. The interval $[0, h_A]_L$ is a logic and all functions from X to P_A vanishing at $X - A$ and attaining constant values on A form a stateless sublogic of $[0, h_A]_L$. Thus, each state on L vanishes at $[0, h_A]_L$ for all $A \in I$. The ideal I of $\mathscr{C}(L)$ is the kernel of each state on L. The factorization (Kalmbach, 1983) of L over I gives Q_z. This factorization induces naturally an affine homeomorphism between $\mathscr{S}(L)$ and $\mathscr{S}(Q_z)$.

To check the automorphism group, we shall show that we can "reconstruct" the factors of the product $\prod_{x \in X} Q_x$ from the structure of the logic L. Each $x \in X$ corresponds to a maximal ideal, say I_x, of $\mathscr{C}(L)$. The logic Q_x is the result of the factorization L/I_x. Suppose that $\alpha \in \mathscr{A}(L)$. The restriction $\alpha|\mathscr{C}(L)$ maps maximal ideals [of $\mathscr{C}(L)$] onto maximal ideals. Hence, it is induced by a pointwise mapping of X onto X. In view of the fact that the logics Q_x, $x \in X$, are mutually nonisomorphic, we conclude that $\alpha|\mathscr{C}(L)$ is the identity. For each $x \in X$, α induces canonically an automorphism α/I_x of $L/I_x = Q_x$. For $x \neq z$, α/I_x coincides with the identity, and $\alpha/I_z \in \mathscr{A}(Q_z)$. Thus, $\mathscr{A}(L)$ is a subgroup of $\mathscr{A}(Q_z) = \mathscr{A}(H)$. On the other hand, suppose that $\beta \in \mathscr{A}(H)$. The mapping $\alpha : L \to L$ defined by the formulas

$$\alpha(f)(x) = f(x) \qquad \text{for} \quad x \neq z$$
$$\alpha(f)(z) = \beta(f(z))$$

is an automorphism of L and $\beta = \alpha/I_z$. We have proved that $\mathscr{A}(L) \cong \mathscr{A}(H) \cong G$. The proof is complete. ■

4. FINAL REMARKS AND RELATED QUESTIONS

In all the sources I cite here the logics with given properties were also lattices. It is natural to ask whether the logic L in the Main Theorem may be a lattice provided that K is a lattice logic. We can prove this under the additional assumption that the Boolean algebra B has at least one atom. The atom corresponds to a singleton in the Stone space of B and we take for z this point. Then the logic L_0 appearing in the proof of the Main Theorem has the required properties. Nevertheless, I do not know of a general construction of lattice logics with given centers and automorphism groups.

If we consider σ-logics ($=\sigma$-orthomodular posets) and the spaces of σ-additive states on them, the situation appears to be much more complicated. Partial results were obtained in Navara and Pták (1988).

It should be noted that Trnková (1988) studied also the independence of other groups related to logics, e.g., the group of affine homeomorphisms of the state space, etc.

A reasonable model of a physical system should admit enough states. If it contains "sufficiently many" two-valued states, we can transfer the problem to set-representable logics [concrete logics; see Pták and Wright (1985). For this case the question of the independence of centers and automorphism groups is investigated in Navara and Tkadlec (1991).

ACKNOWLEDGMENTS

The author would like to express his thanks to P. Pták and V. Rogalewicz for their interest in this research and to V. Trnková for fruitful discussions on this topic.

REFERENCES

Binder, J. (1986). *Reports on Mathematical Physics*, **24**, 73-77.
Brabec, J., and Pták, P. (1982). *Foundations of Physics*, **12**, 107-112.
Gleason, A. (1957). *Journal of Mathematics and Mechanics*, **6**, 428-442.
Greechie, R. J. (1971). *Journal of Combinatorial Theory*, **10**, 119-132.
Gudder, S. P. (1979). *Stochastic Methods in Quantum Mechanics*, North-Holland, New York.
Kallus, M., and Trnková, V. (1987). *International Journal of Theoretical Physics*, **26**, 1-9.
Kalmbach, G. (1983). *Orthomodular Lattices*, Academic Press, London.
Kalmbach, G. (1984). *Bulletin of the Australian Mathematical Society*, **29**, 309-313.
Marlow, A. R. (1978). *Journal of Mathematical Physics*, **19**, 1-15.
Navara, M., and Pták, P. (1988). Enlargements of logics (σ-orthocomplete case), in *Proceedings Conference on Topology and Measure V*, Griefswald, pp. 109-115.
Navara, M., and Rogalewicz, V. (1991). *Mathematische Nachrichten*, **154**, 157-168.
Navara, M., and Tkadlec, J. (1991). *Comment. Math. Univ. Carolin.*, **32**, 15-25.
Navara, M., Pták, P., and Rogalewicz, V. (1988). *Pacific Journal of Mathematics*, **135**, 361-369.
Pták, P. (1983). *Proceedings of the American Mathematical Society*, **88**, 106-109.
Pták, P. (1987). *Colloquia Mathematica*, **54**, 1-7.
Pták, P., and Wright, J. D. M. (1985). *Aplikace Matematiky*, **30**, 274-285.
Sabidussi, G. (1960). *Monatshafte für Mathematik*, **64**, 64-67.
Shultz, F. W. (1974). *Journal of Combinatorial Theory* (A), **17**, 317-328.
Trnková, V. (1988). Symmetries and state automorphisms of quantum logics, in *Proceedings 1st Winter School on Measure Theory, Liptovský Ján*, pp. 170-175.